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THE COTYPE AND UNIFORM CONVEXITY OF UNITARY IDEALS

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ABSTRACT

Information about geometric properties, such as uniform convexity and smoothness, type and cotype, of a unitary Banach ideal S_E is obtained from properties of the symmetric Banach sequence space E. In particular S_E has cotype 2 if E does. The proofs use real interpolation and complex geometry.

1. Introduction

In this paper we solve some problems which arise in the theory of unitary ideals. There is a natural correspondence between unitary ideals and symmetric Banach sequence spaces, and one would expect properties of an unitary ideal to reflect the properties of the corresponding sequence space. Thus the first-named author showed [5] that topological properties are related in a natural way, and the second-named author showed [12] that the ideals S_p have the same convexity and smoothness, type and cotype properties as the spaces l_p. Other geometric and topological properties have been investigated by Arazy ([1], [2]).

Nevertheless, the lack of commutativity leads to non-trivial difficulties, and the problems which we consider have been in circulation for at least a decade.

In this paper, we make systematic use of real interpolation methods; this means that we have to renorm the spaces which we consider with equivalent norms. This is not important in considering cotype, which is an isomorphic invariant; we do not know, however, whether the results which we obtain concerning uniform convexity (which is an isometric invariant) are valid without renorming.

In considering unitary ideals, it is of course natural to consider ideals of operators on a *complex* Hilbert space, and we also make fundamental use of the

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complex geometry of the spaces in question. Thus uniform PL-convexity, which is implied by (real) uniform convexity and which implies cotype properties, turns out to be a very useful tool. A detailed study of the properties of uniform PL-convexity is made elsewhere [4]. Let us give the definitions and mention the basic result that we use. Suppose that $(E,\|\ \|)$ is a complex Banach space and that $0<p<\infty$. We define

$$H_p(\varepsilon)=\inf\left\{\left(\frac{1}{2\pi}\int_0^{2\pi}\|x+e^{i\theta}y\|^p d\theta\right)^{1/p}-1:\|x\|=1,\|y\|=\varepsilon\right\}.$$

$(E,\|\ \|)$ is *q-uniformly* PL-*convex* (where $2\leqq q<\infty$) if there exists $k>0$ such that $H_1(\varepsilon)\geqq k\varepsilon^q$ for $0<\varepsilon\leqq 1$. If $(E,\|\ \|)$ is q-uniformly PL-convex, it follows that E has cotype q.

Let us now describe the structure of this paper. In section 2, we describe the interpolation methods that we use; these are related to the real K-methods of classical interpolation theory. We are in fact only concerned with interpolation between sequence spaces, and between unitary ideals. In Section 3 we establish some operator norm inequalities; although we shall need them later on, they appear to have some independent interest. In the next section we introduce the concept of K-monotonic norms for sequence spaces and establish the relationship between such norms and the uniform PL-convexity and (real) uniform convexity of the corresponding unitary ideals. In Section 5 we show that a p-convex symmetric Banach sequence space can be given an equivalent K-p-monotonic norm. This leads to results concerning cotype. In the final section, we introduce a different renorming for a symmetric Banach sequence space which satisfies an upper p-estimate and a lower q-estimate. This ensures the uniform convexity and uniform smoothness of the corresponding unitary ideal with good control of the moduli. Here there are more properties to be preserved, and the renorming is correspondingly more complicated.

Throughout the paper we use standard Banach space theory notation. We refer to [8] for definitions and notation from the theory of Banach lattices, and to [6] for the basic facts about ideals of operators acting in a Hilbert space.

If A is a compact operator acting in a Hilbert space then $|A|$ denotes the "modulus of A", $|A|=\sqrt{A^*A}$, and $s(A)=\{s_j(A)\}_{j=1}^\infty$ denotes the sequence of singular numbers of A, i.e., $s_j(A)$ is the jth eigenvalue of $|A|$ (where eigenvalues are counted in non-increasing order, according to their multiplicity). Suppose that $(E,\|\ \|)$ is a symmetric Banach sequence space. The corresponding unitary ideal S_E is the space

$$S_E=\{A \text{ compact: } s(A)\in E\},$$

with the norm $\|A\|_E = \|s(A)\|$ for $A \in S_E$.

In the case $E = l_p$, for $1 \leqq p < \infty$, we use the notation $S_{l_p} = S_p$ and $\|A\|_p = (\sum_{j=1}^{\infty} s_j(A)^p)^{1/p}$ for $A \in S_p$. Finally, S_∞ denotes the space of all compact operators and $\|.\|_\infty$ the usual operator norm.

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2. Some interpolation norms

In this section we consider some interpolation norms related to the real K-method ([3] chapters 3 and 5). Suppose that (X_0, X_1) is a couple of Banach spaces. Recall ([3] p. 38) that if $x \in X_0 + X_1$ and $\tau > 0$, $K(\tau, x; X_0, X_1)$ is defined by

$$K(\tau, x; X_0, X_1) = \inf\{\|x_0\| + \tau\|x_1\| : x_0 \in X_0,\ x_1 \in X_1,\ x = x_0 + x_1\}.$$

For our purposes, it will be more convenient to work with the quantity

$$K_2(\tau, x; X_0, X_1) = \inf\{(\|x_0\|^2 + \tau^2\|x_1\|^2)^{\frac{1}{2}} : x_0 \in X_0,\ x_1 \in X_1,\ x = x_0 + x_1\}.$$

Clearly

$$K_2(\tau, x; X_0, X_1) \leqq K(\tau, x; X_0, X_1) \leqq \sqrt{2}\, K_2(\tau, x, X_0, X_1).$$

Let $1 \leqq r < s \leqq \infty$. It is easy to see that

$$K_2(\tau, x; l_r, l_s) = K_2(\tau, |x|; l_r, l_s) = K_2(\tau, x^*; l_r, l_s),$$

for $x \in l_s$ and all $\tau > 0$, and that

$$K_2(\tau, x; l_r, l_s) = \inf\{(\|x_0\|_r^2 + \tau^2\|x_1\|_s^2)^{\frac{1}{2}} : x_0 \in l_r,\ x_1 \in l_s,\ x_0, x_1 \geqq 0,\ x = x_0 + x_1\}$$

for all $\tau > 0$, if x is a positive element of l_s.

The next proposition, which will play a fundamental role in our arguments, follows from the corresponding results for K ([7] theorem 4.1 and [3] theorem 5.2.1; note though that there are errors in [3] exercises 5.7.2 and 5.7.3).

PROPOSITION 1. *Suppose that $1 \leqq r < s < \infty$. There exist positive constants c_1 and c_2 such that*

$$c_2K_2(m^\alpha, x; l_r, l_s) \leqq \left(\sum_{j=1}^m x_j^{*r}\right)^{1/r} + m^\alpha \left(\sum_{j=m+1}^\infty x_j^{*s}\right)^{1/s}$$
$$\leqq c_1K_2(m^\alpha, x; l_r, l_s),$$

for all $x \in l_s$ *and* $m = 1, 2, \ldots,$ *where* $\alpha = 1/r - 1/s$.

Furthermore,

$$\tfrac{1}{2}K_2(m^{1/r}, x; l_r, l_\infty) \leqq \left(\sum_{j=1}^m x_j^{*r}\right)^{1/r} \leqq \sqrt{2}\, K_2(m^{1/r}, x; l_r, l_\infty)$$

for all $x \in l_\infty$ *and* $m = 1, 2, \ldots$.

We now establish some convexity and concavity properties of K_2.

Proposition 2. *Let* $1 \leqq r \leqq 2 \leqq q \leqq \infty$. *If* x *and* y *are in* l_q, *and* $\tau > 0$, *then*

$$K_2(\tau, (|x|^r + |y|^r)^{1/r}; l_r, l_q)^r \leqq K_2(\tau, x; l_r, l_q)^r + K_2(\tau, y; l_r, l_q)^r$$

and

$$K_2(\tau, (|x|^q + |y|^q)^{1/q}; l_r, l_q)^q \geqq K_2(\tau, x; l_r, l_q)^q + K_2(\tau, y; l_r, l_q)^q.$$

Proof. Throughout this proof we shall write $K_2(\tau, x)$ for $K_2(\tau, x; l_r, l_q)$. To show the first inequality suppose that $|x| = x_0 + x_1$ and $|y| = y_0 + y_1$ with x_0 and y_0 positive elements of l_r and x_1 and y_1 positive elements of l_q. Then

$$K_2(\tau, (|x|^r + |y|^r)^{1/r}) \leqq K_2(\tau, (x_0^r + y_0^r)^{1/r} + (x_1^r + y_1^r)^{1/r})$$
$$\leqq (\|(x_0^r + y_0^r)^{1/r}\|_r^2 + \tau^2 \|(x_1^r + y_1^r)^{1/r}\|_q^2)^{1/2}$$
$$\leqq [(\|x_0\|_r^r + \|y_0\|_r^r)^{2/r} + \tau^2(\|x_1\|_q^r + \|y_1\|_q^r)^{2/r}]^{1/2},$$

since l_r and l_q are both r-convex with constants equal to 1. Since $2/r \geqq 1$, by the triangle inequality for the space $l_{2/r}$ the last expression is less than or equal to

$$[(\|x_0\|_r^2 + \tau^2 \|x_1\|_q^2)^{r/2} + (\|y_0\|_r^2 + \tau^2 \|y_1\|_q^2)^{r/2}]^{1/r}.$$

Taking the infimum over all such decompositions of $|x|$ and $|y|$, it follows that

$$K_2(\tau, (|x|^r + |y|^r)^{1/r}) \leqq (K_2(\tau, |x|)^r + K_2(\tau, |y|)^r)^{1/r}$$
$$= (K_2(\tau, x)^r + K_2(\tau, y)^r)^{1/r}.$$

To show the second inequality suppose that $(|x|^q + |y|^q)^{1/q} = z_0 + z_1$, with z_0 a positive element of l_r and z_1 a positive element of l_q. Define elements

$$x_0 = z_0 x(|x|^q + |y|^q)^{-1/q}, \qquad x_1 = z_1 x(|x|^q + |y|^q)^{-1/q}$$

and

$$y_0 = z_0 y(|x|^q + |y|^q)^{-1/q}, \qquad y_1 = z_1 y(|x|^q + |y|^q)^{-1/q}.$$

Note that $x = x_0 + x_1$ and $y = y_0 + y_1$. Moreover, $z_0 = (|x_0|^q + |y_0|^q)^{1/q}$ and $z_1 = (|x_1|^q + |y_1|^q)^{1/q}$.

Therefore, since both l_r and l_q are q-concave with constants equal to 1 and since $q/2 \geqq 1$,

$$\begin{aligned}(\|z_0\|_r^2 + \tau^2\|z_1\|_q^2)^{1/2} &\geqq [(\|x_0\|_r^q + \|y_0\|_r^q)^{2/q} + \tau^2(\|x_1\|_q^q + \|y_1\|_q^q)^{2/q}]^{1/2} \\ &\geqq [(\|x_0\|_r^2 + \tau^2\|x_1\|_q^2)^{q/2} + (\|y_0\|_r^2 + \tau^2\|y_1\|_q^2)^{q/2}]^{1/q} \\ &\geqq (K_2(\tau, x)^q + K_2(\tau, y)^q)^{1/q}.\end{aligned}$$

Taking the infimum over all such decompositions of $(|x|^q + |y|^q)^{1/q}$, we obtain that

$$K(\tau, (|x|^q + |y|^q)^{1/q}) \geqq [K_2(\tau, x)^q + K_2(\tau, y)^q]^{1/q}.$$

In the final proposition of this section, we consider interpolation between unitary ideals.

PROPOSITION 3. *Let* $1 \leqq r < s \leqq \infty$. *Suppose that* $A \in S_s$. *Then*

(i) $K_2(\tau, A; S_r, S_s) = K_2(\tau, s(A); l_r, l_s)$ *for all* $\tau > 0$.

(ii) *If* A *is Hermitian,*

$$\begin{aligned}K_2(\tau, A; S_r, S_s) = \inf\{(\|A_0\|_r^2 + \tau^2\|A_1\|_s^2)^{1/2} : A_0 \in S_r,\ A_1 \in S_s,\\ A = A_0 + A_1 \text{ and } A_0,\ A_1 \text{ are Hermitian}\}\end{aligned}$$

for all $\tau > 0$.

(iii) *If* A *is positive,*

$$\begin{aligned}K_2(\tau, A; S_r, S_s) = \inf\{(\|A_0\|_r^2 + \tau^2\|A_1\|_s^2)^{1/2} : A_0 \in S_r,\ A_1 \in S_s,\\ A = A_0 + A_1 \text{ and } A_0,\ A_1 \text{ are positive}\}\end{aligned}$$

for all $\tau > 0$.

PROOF. Since $K_2(\tau, A; S_r, S_s) = K_2(\tau, |A|; S_r, S_s)$ for all $\tau > 0$, we can assume in case (i) that A is Hermitian.

Suppose that $A = \sum_{j=1}^{\infty} \varepsilon_j s_j(A)\phi_j \otimes \phi_j$ is Hermitian (where $\varepsilon_j = \pm 1$) and that $A = A_0 + A_1$, with A_0 in S_r and $A_1 \in S_s$. Let

$$a_j^{(0)} = \langle A_0\phi_j, \phi_j\rangle, \qquad a_j^{(1)} = \langle A_1\phi_j, \phi_j\rangle,$$

for $j = 1, 2, \dots$ and let

$$A_0' = \sum_{j=1}^{\infty} a_j^{(0)} \phi_j \otimes \phi_j, \qquad A_1' = \sum_{j=1}^{\infty} a_j^{(1)} \phi_j \otimes \phi_j.$$

Then $A_0' \in S_r$, $A_1' \in S_s$,

$$\|A_0'\|_r = \|(a_j^{(0)})\|_r \leqq \|A_0\|_r, \qquad \|A_1'\|_s = \|(a_j^{(1)})\|_s \leqq \|A_1\|_s,$$

$A = A_0' + A_1'$ and $(\varepsilon_j s_j(A)) = (a_j^{(0)}) + (a_j^{(1)})$. Thus

$$\begin{aligned} K_2(\tau, A; S_r, S_s) &= \inf\Big\{(\|A_0\|_r^2 + \tau^2\|A_1\|_s^2)^{1/2} : A_0 = \sum_{j=1}^{\infty} a_j^{(0)}\phi_j \otimes \phi_j \in S_s, \\ &\qquad A_1 = \sum_{j=1}^{\infty} a_j^{(1)}\phi_j \otimes \phi_j \in S_s \text{ and } A = A_0 + A_1\Big\} \\ &= \inf\{(\|a_0\|_r^2 + \tau^2\|a_1\|_s^2)^{1/2} : a_0 \in l_r,\ a_1 \in l_s \text{ and} \\ &\qquad (\varepsilon_j s_j(A)) = a_0 + a_1\} \\ &= \inf\{(\|a_0\|_r^2 + \tau^2\|a_1\|_s^2)^{1/2} : a_0 \in l_r,\ a_1 \in l_s, \\ &\qquad a_0, a_1 \geqq 0 \text{ and } s(A) = a_0 + a_1\}. \end{aligned}$$

The first equality gives (ii), the second gives (i) and the third equality combined with the first gives (iii). Part (i) of this proposition was proved by Arazy ([1] theorem 2.4).

3. Some operator norm inequalities

In this section we shall prove some operator norm inequalities which we shall need to establish convexity properties of unitary ideals. The first set of inequalities is quite general.

PROPOSITION 4. *Suppose that* $1 < p < \infty$ *and* $1 < u < \infty$. *There exist positive constants* $c_{p,u}$ *and* $C_{p,u}$ *such that if* A *and* B *are operators in* S_p *then*

(i) *if* $1 < p \leqq 2$

$$\begin{aligned}(\|A\|_p^2 + c_{p,u}\|B\|_p^2)^{1/2} &\leqq \{\tfrac{1}{2}(\|A+B\|_p^u + \|A-B\|_p^u)\}^{1/u} \\ &\leqq (\|A\|_p^p + C_{p,u}\|B\|_p^p)^{1/p}\end{aligned}$$

(ii) *while if* $2 \leqq p < \infty$

$$\begin{aligned}(\|A\|_p^p + c_{p,u}\|B\|_p^p)^{1/p} &\leqq \{\tfrac{1}{2}(\|A+B\|_p^u + \|A-B\|_p^u)\}^{1/u} \\ &\leqq (\|A\|_p^2 + C_{p,u}\|B\|_p^2)^{1/2}.\end{aligned}$$

PROOF. The modulus of convexity δ_{S_p} satisfies

$$\delta_{S_p}(\varepsilon) \overset{\circ}{\sim} \varepsilon^{\max(2,p)}$$

by [12], theorem 2.2. The left-hand inequalities follow from this, and proposition 1 of [4].

The right-hand inequalities are established by duality. Suppose that $p \geqq 2$. Let $q = p/(p-1)$ and $v = u/(u-1)$ be the conjugate indices, and let $\lambda = \|A+B\|_p$ and $\mu = \|A-B\|_p$. There exist operators S and T in S_q such that $\|S\|_q = \|T\|_q = 1$ and

$$\operatorname{Tr}(A+B)S = \lambda, \qquad \operatorname{Tr}(A-B)T = \mu.$$

Then

$$\begin{aligned}
\tfrac{1}{2}(\lambda^u + \mu^u) &= \tfrac{1}{2}(\lambda^{u-1}\operatorname{Tr}(A+B)S + \mu^{u-1}\operatorname{Tr}(A-B)T) \\
&= \tfrac{1}{2}(\operatorname{Tr} A(\lambda^{u-1}S + \mu^{u-1}T) + \operatorname{Tr} B(\lambda^{u-1}S - \mu^{u-1}T)) \\
&\leqq \tfrac{1}{2}(\|A\|_p^2 + c_{q,v}^{-1}\|B\|_p^2)^{1/2}(\|\lambda^{u-1}S + \mu^{u-1}T\|_q^2 + c_{q,v}\|\lambda^{u-1}S - \mu^{u-1}T\|_q^2)^{1/2} \\
&\leqq (\|A\|_p^2 + c_{q,v}^{-1}\|B\|_p^2)^{1/2}(\tfrac{1}{2}(\lambda^{(u-1)v} + \mu^{(u-1)v}))^{1/v}
\end{aligned}$$

by the left-hand inequality of (i). Thus we can take $C_{p,u} = c_{q,v}^{-1}$. The case where $1 < p \leqq 2$ is proved in exactly the same way.

The next proposition is concerned with Hermitian operators.

PROPOSITION 5. (i) *If $1 < p \leqq 2$ there exists a positive constant c_p such that if A and B are Hermitian operators in S_p then*

$$(\|A\|_p^2 + c_p\|B\|_p^2)^{1/2} \leqq \|A+iB\|_p \leqq (\|A\|_p^p + \|B\|_p^p)^{1/p}.$$

(ii) *If $2 \leqq p < \infty$ there exists a positive constant C_p such that if A and B are Hermitian operators in S_p then*

$$(\|A\|_p^p + \|B\|_p^p)^{1/p} \leqq \|A+iB\|_p \leqq (\|A\|_p^p + C_p\|B\|_p^p)^{1/p}.$$

PROOF. We shall prove (i); (ii) follows by duality, as in Proposition 4. Since $\|A+iB\|_p = \|A-iB\|_p$ when A and B are Hermitian, the left-hand inequality of (i) follows directly from Proposition 4. We prove the right-hand inequality by interpolation. If A and B are in S_1, then $\|A+iB\|_1 \leqq \|A\|_1 + \|B\|_1$, by the triangle inequality. If $p = 2$, and A and B are Hermitian

$$\begin{aligned}
\|A+iB\|_2^2 &= \operatorname{tr}((A+iB)(A-iB)) \\
&= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) + i\operatorname{tr}(BA) - i\operatorname{tr}(AB) \\
&= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) = \|A\|_2^2 + \|B\|_2^2
\end{aligned}$$

(cf. [6] theorem 8.2).

Suppose now that $1<p<2$, that A and B are Hermitian elements of S_p, and that

$$A=\sum_{j=1}^{\infty}\alpha_j\phi_j\otimes\phi_j,\qquad B=\sum_{j=1}^{\infty}\beta_j\psi_j\otimes\psi_j.$$

Define an operator $T: l_2(\mathbf{R})\otimes_2 l_2(\mathbf{R})\rightarrow S_2$ by

$$T((\gamma_j),(\delta_j))=\sum_{j=1}^{\infty}\gamma_j\phi_j\otimes\phi_j+i\sum_{j=1}^{\infty}\delta_j\psi_j\otimes\psi_j.$$

Considering S_2 as a real linear space, T is a real linear mapping and $\|T\|=1$. Also

$$\|T: l_1(\mathbf{R})\oplus_1 l_1(\mathbf{R})\rightarrow S_1\|\leqq 1.$$

Now by [3] theorem 5.2.2, the quasi-normed space $(l_p(\mathbf{R})\oplus_p l_p(\mathbf{R}))^p$ is, up to a multiplicative constant, isometric to the real interpolation space

$$(l_1(\mathbf{R})\oplus_1 l_1(\mathbf{R}),(l_2(\mathbf{R})\oplus_2 l_2(\mathbf{R}))^2)_{p-1,1}$$

and, arguing as in Proposition 3, it is easy to see that $(S_p)^p$ is, up to the same multiplicative constant, isometric to $(S_1,(S_2)^2)_{p-1,1}$. By real interpolation ([3] theorem 3.11.2) it follows that

$$\|T: l_p(\mathbf{R})\oplus_p l_p(\mathbf{R})\rightarrow S_p\|\leqq 1.$$

Applying T to $((\alpha_j),(\beta_j))$, it follows that

$$\|A+iB\|_p^p\leqq\|A\|_p^p+\|B\|_p^p.$$

The right-hand inequality in (i) and the left-hand inequality in (ii) also follow directly from Clarkson's inequality for the spaces S_p [9].

The final results of this section involve positive operators.

Proposition 6. (i) *If A is a positive operator and B a Hermitian operator in $L(H)$ then*

$$\|A+iB\|\leqq(\|A\|^2+2\|B\|^2)^{1/2}.$$

(ii) *If A is a positive element of S_1 and B a Hermitian element of S_1 then*

$$\|A+iB\|_1\geqq(\|A\|_1^2+\tfrac{1}{2}\|B\|_1^2)^{1/2}.$$

Proof. First observe that it is sufficient to consider the case where H is two dimensional. For given $\eta>0$ there exists a unit vector h in H such that

$\|(A+iB)h\| \geqq (1-\eta)\|(A+iB)\|$. Let P be the orthogonal projection onto span $(h,(A+iB)h)$. Then

$$\|PAP+iPBP\| \geqq (1-\eta)\|A+iB\|,$$

while PAP is positive, PBP is Hermitian, and $\|PAP\| \leqq \|A\|$, $\|PBP\| \leqq \|B\|$.

By choosing a suitable basis, we can suppose that A and B have matrix representations

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad B = \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix}$$

where $\|A\| = \lambda_1 \geqq \lambda_2$. Let

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}.$$

Note that $\|B_1\| \leqq \|B\|$ and $\|B_2\| = |c| \leqq \|B\|$. Since $AB_1 = B_1A$,

$$(A-iB)(A+iB) = A^2 + B^2 + i(AB_2 - B_2A)$$

so that

$$\|A+iB\|^2 = \|(A-iB)(A+iB)\| \leqq \|A^2 + i(AB_2 - B_2A)\| + \|B^2\|.$$

Now $A^2 + i(AB_2 - B_2A)$ has matrix representation

$$\begin{pmatrix} \lambda_1^2 & i(\lambda_1-\lambda_2)c \\ -i(\lambda_1-\lambda_2)\bar{c} & \lambda_2^2 \end{pmatrix}.$$

If $\lambda_1 = \lambda_2$, this has norm $\lambda_1^2 = \|A\|^2$. Otherwise, the norm is equal to the larger eigenvalue; a simple calculation shows that

$$\begin{aligned}\|A^2 + i(AB_2 - B_2A)\| &= \tfrac{1}{2}\left(\lambda_1^2 + \lambda_2^2 + \sqrt{(\lambda_1^2-\lambda_2^2)^2 + 4(\lambda_1-\lambda_2)^2 c\bar{c}}\right) \\ &= \lambda_1^2 + \tfrac{1}{2}(\lambda_1^2 - \lambda_2^2)\left(\sqrt{1+\frac{4c\bar{c}}{(\lambda_1+\lambda_2)^2}} - 1\right) \\ &\leqq \lambda_1^2 + c\bar{c}\frac{(\lambda_1^2-\lambda_2^2)}{(\lambda_1+\lambda_2)^2} \\ &= \lambda_1^2 + c\bar{c}\left(\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right) \leqq \|A\|^2 + \|B\|^2.\end{aligned}$$

The result for S_1 follows by duality.

REMARK. Operators represented by matrices

$$A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}0 & t\\ t & 0\end{pmatrix}$$

for small real values of t show that 2 is the best possible constant in the first inequality of this proposition.

If A and B are represented by

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}0 & it\\ -it & 0\end{pmatrix},$$

$\|A+iB\|=1+t=\|A\|+\|B\|$; thus it is not possible to drop the condition that A is positive.

4. K-(r,s) monotonicity and the convexity of unitary ideals

We now make some definitions relating the K_2-functionals to convexity properties of a sequence space. Suppose that $(E,\|.\|)$ is a Banach sequence space. If $1<r\leqq s<\infty$ we say that the norm is *K-(r,s)-monotonic* if, whenever v, x, y, z are vectors in E which satisfy

$$(K_2(\tau,v;l_r,l_s)^s+K_2(\tau,x;l_r,l_s)^s)^{1/s}\leqq(K_2(\tau,y;l_r,l_s)^r+K_2(\tau,z;l_r,l_s)^r)^{1/r}$$

for all $\tau>0$, it follows that

$$(\|v\|^s+\|x\|^s)^{1/s}\leqq(\|y\|^r+\|z\|^r)^{1/r}.$$

If $1<p<\infty$ we say that the norm is *K-(p,∞)-monotonic* (*K-p-monotonic*, for short) if, whenever x, y and z are vectors in E which satisfy

$$K_2(\tau,x;l_p,l_\infty)^p\leqq K_2(\tau,y;l_p,l_\infty)^p+K_2(\tau,z;l_p,l_\infty)^p$$

for all $\tau>0$, it follows that

$$\|x\|^p\leqq\|y\|^p+\|z\|^p.$$

The relevance of these definitions is suggested by the following proposition, which is an immediate consequence of Proposition 2 and the definitions of r-convexity and s-concavity.

PROPOSITION 7. *Suppose that $1\leqq r\leqq 2\leqq s\leqq\infty$. If $(E,\|\ \|)$ is a Banach sequence space with a K-(r,s)-monotonic norm then $(E,\|\ \|)$ is symmetric, and is r-convex and s-concave, with $M^{(r)}(E)=M_{(s)}(E)=1$.*

The next result shows that K-(r,s) monotonicity leads to important consequences for unitary ideals.

PROPOSITION 8. *Suppose that* $(E, \|.\|)$ *is a symmetric Banach sequence space.*

(i) *If* $1 < p \leqq 2$ *and* $\|.\|$ *is* K-p-*monotonic, and if* A *is a positive operator and* B *a Hermitian operator in* S_E, *then*

$$\|A + iB\|_E^p \leqq \|A\|_E^p + 2\|B\|_E^p.$$

(ii) *If* $1 < r \leqq 2 \leqq s < \infty$ *there exist positive constants* c *and* C *such that if* $\|.\|$ *is a* K-(r,s)-*monotonic norm then*

$$[\tfrac{1}{2}(\|A + B\|_E^r + \|A - B\|_E^r)]^{1/r} \geqq (\|A\|_E^s + c\|B\|_E^s)^{1/s}$$

and

$$[\tfrac{1}{2}(\|A + B\|_E^s + \|A - B\|_E^s)]^{1/s} \leqq (\|A\|_E^r + C\|B\|_E^r)^{1/r}$$

for all operators A *and* B *in* S_E.

PROOF. (i) Suppose that $A = A_0 + A_1$, where A_0 and A_1 are positive and that $B = B_0 + B_1$, where B_0 and B_1 are Hermitian. Then

$$A + iB = (A_0 + iB_0) + (A_1 + iB_1),$$

and so, if $\tau > 0$,

$$\begin{aligned} K_2(\tau, A + iB; S_p, S_\infty) &\leqq (\|A_0 + iB_0\|_p^2 + \tau^2\|A_1 + iB_1\|_\infty^2)^{1/2} \\ &\leqq ((\|A_0\|_p^p + 2\|B_0\|_p^p)^{2/p} + (\tau^p\|A_1\|_\infty^p + 2\tau^p\|B_1\|_\infty^p)^{2/p})^{1/2} \end{aligned}$$

by Propositions 5 (i) and 6 (i). By the triangle inequality in $l_{2/p}$, it follows that

$$K_2(\tau, A + iB; S_p, S_\infty) \leqq [(\|A_0\|_p^2 + \tau^2\|A_1\|_p^2)^{p/2} + 2(\|B_0\|_p^2 + \tau^2\|B_1\|_p^2)^{p/2}]^{1/p}.$$

Taking the infimum over all such decompositions of A and B, it follows that

$$K_2(\tau, A + iB; S_p, S_\infty)^p \leqq (K_2(\tau, A; S_p, S_\infty))^p + 2(K_2(\tau, B; S_p, S_\infty))^p.$$

Thus

$$K_2(\tau, s(A + iB); l_p, l_\infty)^p \leqq (K_2(\tau, s(A); l_p, l_\infty))^p + 2(K_2(\tau, s(B); l_p, l_\infty))^p$$

so that

$$\|A + iB\|_E^p = \|s(A + iB)\|^p \leqq \|s(A)\|^p + 2\|s(B)\|^p = \|A\|_E^p + 2\|B\|_E^p$$

by the K-p-monotonicity of the norm.

(ii) To show the first inequality suppose that $A + B = T_0 + T_1$ and $A - B = R_0 + R_1$, where T_0 and R_0 are in S_r and T_1 and R_1 are in S_s. Let $A_0 = \tfrac{1}{2}(T_0 + R_0)$ and $A_1 = \tfrac{1}{2}(T_1 + R_1)$ and, similarly, let $B_0 = \tfrac{1}{2}(T_0 - R_0)$ and $B_1 = \tfrac{1}{2}(T_1 - R_1)$. Note that $A = A_0 + A_1$ and $B = B_0 + B_1$. So if $\tau > 0$ and $c > 0$,

$$\begin{aligned}(K_2(\tau, A; S_r, S_s)^s &+ cK_2(\tau, B; S_r, S_s)^s)^{1/s}\\ &\leqq \{(\|A_0\|_r^2 + \tau^2\|A_1\|_s^2)^{s/2} + c(\|B_0\|_r^2 + \tau^2\|B_1\|_s^2)^{s/2}\}^{1/s}\\ &\leqq \{(\|A_0\|_r^s + c\,\|B_0\|_r^s)^{2/s} + \tau^2(\|A_1\|_s^s + c\,\|B_1\|_s^s)^{2/s}\}^{1/2},\end{aligned}$$

by the triangle inequality in $l_{s/2}$. Since $s \geqq 2$, it follows from Proposition 4 that if $c = \min(c_{r,r}^{s/2}, c_{s,r})$ then

$$\begin{aligned}(K_2(\tau, A; S_r, S_s)^s &+ cK_2(\tau, B; S_r, S_s)^s)^{1/s}\\ &\leqq \{(\tfrac12(\|A_0+B_0\|_r^r + \|A_0-B_0\|_r^r))^{2/r} + \tau^2(\tfrac12(\|A_1+B_1\|_s^r + \|A_1-B_1\|_s^r))^{2/r}\}^{1/r}\\ &\leqq \{\tfrac12[(\|A_0+B_0\|_r^2 + \tau^2\|A_1+B_1\|_s^2)^{r/2} + (\|A_0-B_0\|_r^2 + \tau^2\|A_1-B_1\|_s^2)^{r/2}]\}^{1/r}\\ &= \{\tfrac12[(\|T_0\|_r^2 + \tau^2\|T_1\|_s^2)^{r/2} + (\|R_0\|_r^2 + \tau^2\|R_1\|_s^2)^{r/2}]\}^{1/r},\end{aligned}$$

by the triangle inequality in $l_{2/r}$. Taking the infimum over all such decompositions of $A+B$ and $A-B$, it follows that

$$\begin{aligned}(K_2(\tau, A; S_r, S_s)^s &+ cK_2(\tau, B; S_r, S_s)^s)^{1/s}\\ &\leqq \{\tfrac12(K_2(\tau, A+B; S_r, S_s)^r + K_2(\tau, A-B; S_r, S_s)^r)\}^{1/r}.\end{aligned}$$

Thus, by Proposition 3 (i),

$$\begin{aligned}\{K_2(\tau, s(A); l_r, l_s)^s &+ cK_2(\tau, s(B); l_r, l_s)^s\}^{1/s}\\ &\leqq \{\tfrac12(K_2(\tau, s(A+B); l_r, l_s)^r + K_2(\tau, s(A-B); l_r, l_s)^r)\}^{1/r},\end{aligned}$$

and so the first inequality follows by K-(r,s)-monotonicity of the norm.

The proof of the second inequality is analogous, and we omit it.

THEOREM 1. *Suppose that $1 < p \leqq 2$ and that E is a symmetric Banach sequence space, whose dual E' has a K-p-monotonic norm. Then the associated unitary ideal S_E is p'-uniformly* PL-*convex* (*where* $1/p + 1/p' = 1$).

PROOF. Suppose that S and T are in S_E, with $\|S\| = 1$. There exist partial isometries U and V, with U onto and V one-one, such that VSU is positive. Suppose that $\varepsilon > 0$. There exist unit vectors A and B in $S_{E'}$, with A positive, such that

$$\operatorname{tr}(VSUA) \geqq 1 - \varepsilon, \qquad |\operatorname{tr}(VTUB) - \|T\|| < \varepsilon\,\|T\|.$$

Let $C_\theta = ie^{i\theta}B^* - ie^{-i\theta}B$, for $0 < \theta \leqq 2\pi$. Note that C_θ is Hermitian, so that if $\beta > 0$

$$\|A + i\beta C_\theta\|_E^p \leqq 1 + 2(2\beta)^p,$$

by Proposition 8(i). Thus, since $\|S + e^{i\theta}T\| = \|VSU + e^{i\theta}VTU\|$,

$$(1+2(2\beta)^p)^{1/p}\frac{1}{2\pi}\int_0^{2\pi}\|(S+e^{i\theta}T)\|\,d\theta$$

$$\geqq \operatorname{Re}\frac{1}{2\pi}\int_0^{2\pi}\operatorname{tr}(VSU+e^{i\theta}VTU)(A+i\beta C_\theta)\,d\theta$$

$$= \operatorname{Re}\operatorname{tr}(VSUA+\beta VTUB)$$

$$\geqq (1-\varepsilon)(1+\beta\|T\|).$$

Setting $\beta = \|T\|^{p'-1}/2^{2p'-1}$, it follows, since ε is arbitrary, that

$$\frac{1}{2\pi}\int_0^{2\pi}\|S+e^{i\theta}T\|\,d\theta \geqq \left(1+\frac{\|T\|^{p'}}{2^{2p'-1}}\right)^{1/p'}.$$

THEOREM 2. *Suppose that $1<r\leqq 2\leqq s<\infty$ and that $(E,\|.\|)$ is a symmetric Banach sequence space whose norm is K-(r,s)-monotonic. Then S_E is uniformly convex with modulus of convexity of power type s and uniformly smooth with modulus of smoothness of power type r.*

PROOF. Suppose that $\varepsilon>0$ and that T and R are in S_E, with $\|T\|_E = \|R\|_E = 1$ and $\|T-R\|_E = \varepsilon$. It follows from the first inequality in Proposition 8(ii), applied to $A=(T+R)/2$ and $B=(T-R)/2$, that

$$\left(\left\|\frac{T+R}{2}\right\|_E^s + c\left(\frac{\varepsilon}{2}\right)^s\right)^{1/s}\leqq 1,$$

where $c>0$ depends only on r and s. Therefore

$$1-\left\|\frac{T+R}{2}\right\|_E \geqq 1-\left(1-c\left(\frac{\varepsilon}{2}\right)^s\right)^{1/s}\geqq c'\varepsilon^s.$$

Taking the infimum over all such operators T and R, it follows that $\delta_{s_E}(\varepsilon)\geqq C'\varepsilon^s$.

Next suppose that $\tau>0$ and T and R are in S_E, with $\|T\|_E=1$, $\|R\|_E=\tau$. From the second inequality in Proposition 8(ii) it follows that

$$\{\tfrac{1}{2}(\|T+R\|_E^s+\|T-R\|_E^s)\}^{1/s}\leqq(1+C\tau^r)^{1/r}.$$

Thus

$$\tfrac{1}{2}(\|T+R\|_E+\|T-R\|_E)-1\leqq(1+C\tau^r)^{1/r}-1\leqq C'\tau^r.$$

Taking the supremum over all such operators T and R it follows that $\rho_{S_E}(\tau)\leqq C'\tau^r$.

5. The cotype of operator ideals

Theorem 1 raises the questions: Under what circumstances does a symmetric Banach sequence space have a K-p-monotonic norm? Under what circumstances can a symmetric Banach sequence space be given an equivalent symmetric norm which is K-p-monotonic? We shall consider the second of these questions. The next proposition is well-known: for completeness' sake we give a proof.

PROPOSITION 9. *Let $(X, \| \ \|)$ be a Köthe function space for which X' is norming, and suppose that $1 < p < \infty$. Then $(X, \| \ \|)$ is p-convex, with $M^{(p)}(X) = 1$, if and only if there exists a set A of non-negative measurable functions such that*

$$\|f\| = \sup_{a \in A} \left(\int |f|^p a d\mu \right)^{1/p}$$

for each f in X.

PROOF. If f is measurable, let $f^p = |f|^p \operatorname{sgn} f$. Let $X^p = \{f^p : f \in X\}$, and if $g = f^p \in X^p$, let $\|g\|_{(p)} = \|f\|^p$. Then $(X^p, \|f\|_{(p)})$ is a Köthe function space, which is a concrete representation of the p-concavification of $(X, \| \ \|)$ ([8] p. 54). If $0 \leqq g_n \uparrow g$ a.e. then $0 \leqq g_n^{1/p} \uparrow g^{1/p}$ a.e., so that, by [8] proposition 1.6.18, $(X^p)'$ is norming for $(X^p, \| \ \|_{(p)})$. Thus there exists a set A of non-negative measurable functions such that

$$\|g\|_{(p)} = \sup_{a \in A} \int |g| a d\mu$$

for each g in X^p. Consequently

$$\|f\| = \|f^p\|_{(p)}^{1/p} = \sup_{a \in A} \left(\int |f|^p a d\mu \right)^{1/p}$$

for each f in X. The converse implication is trivial.

Suppose now that $(X, \| \ \|)$ is a rearrangement invariant space on Ω, in the sense of [8] (so that $\Omega = \{1, 2, \dots\}$, $[0,1]$ or $[0, \infty)$). Then there exists a set A of non-negative non-increasing functions on Ω such that

$$\|f\| = \sup_{a \in A} \int f^* a d\mu$$

(where f^* is the decreasing rearrangement of f), for each f in X. Let us set

$$f^\dagger(t) = \int_0^t f^*(s) d\mu(s),$$

and let M be the set of positive measures $\{-da : a \in A\}$. Then

$$\|f\| = \sup_{\nu \in M} \int_{\Omega} f^{\dagger} d\nu$$

for each f in X. Following through the argument of Proposition 9, we obtain

PROPOSITION 10. *Let $(X, \| \quad \|)$ be a rearrangement invariant space, and suppose that $1 < p < \infty$. Then $(X, \| \quad \|)$ is p-convex, with $M^{(p)}(X) = 1$, if and only if there exists a set M of positive measures on Ω such that*

$$\|f\|^p = \sup_{\nu \in M} \int_{\Omega} (f^p)^{\dagger} d\nu$$

for each f in X.

THEOREM 3. *Suppose that $1 < p \leqq 2$ and that $(E, \| \quad \|)$ is a symmetric Banach sequence space. Then E is p-convex if and only if there exists an equivalent K-p-monotonic norm $\| \quad \|_K$ on E.*

PROOF. Suppose that E is p-convex. By [8] proposition 1.d.8, we can replace the norm on E by an equivalent norm whose p-convexity constant is 1, and this norm is clearly symmetric. By Proposition 10, we can suppose that there exists a set C of positive sequences such that

$$\|x\|^p = \sup_{c \in C} \sum_{n=1}^{\infty} c_n \left(\sum_{j=1}^{n} x_j^{*p} \right)$$

for each x in E. We now set

$$\|x\|_K = \sup_{c \in C} \left(\sum_{n=1}^{\infty} c_n (K_2(n^{1/p}, x; l_p, l_\infty))^p \right)^{1/p}.$$

Clearly $\|\alpha x\|_K = |\alpha| \, \|x\|$. By Proposition 1, $\|x\|_K \leqq \|x\| \leqq 2^{1/p'} \|x\|_K$. It follows from Proposition 2 and the definition of $\| \quad \|_K$ that

$$\|(|x|^p + |y|^p)^{1/p}\|_K \leqq (\|x\|_K^p + \|y\|_K^p)^{1/p},$$

for all x and y in E. Consequently the p-concavification of $(E, \| \quad \|_K)$ is a normed space. $(E \| \quad \|_K)$ is the p-convexification of this, and so is a normed space (cf. [8] p. 53). Thus $\| \quad \|_K$ is a norm on E which is equivalent to $\| \quad \|_E$. Finally the fact that $\| \quad \|_K$ is K-p-monotonic follows directly from the definition of $\| \quad \|_K$.

The converse statement is an immedite consequence of Proposition 2.

Using the duality between q-convexity and q'-concavity, ([8] proposition 1.d.4), the fact that a Banach lattice has cotype 2 if and only if it is 2-concave ([8]

theorem 1.f.16) and the fact that a q-uniformly PL-convex Banach space has cotype q, we obtain the following conclusions.

THEOREM 4. *Suppose that $(E, \| \quad \|)$ is a symmetric Banach sequence space.*

(a) *The following are equivalent*:

(i) *E has cotype* 2;

(ii) *S_E has cotype* 2; *and*

(iii) *there is an equivalent unitary-invariant norm on S_E under which it is* 2-*uniformly* PL-*convex*.

(b) *If E is q-concave (where $2 \leqq q < \infty$), S_E has cotype q, and can be given an equivalent unitary-invariant norm under which it is q-uniformly* PL-*convex*.

Theorem 4, combined with the duality theory for Banach lattices (cf. [8] chapter 1), gives analogous results for the type of a unitary ideal.

COROLLARY. *Suppose that $(E, \| . \|)$ is a symmetric Banach sequence space.*

(a) *The following are equivalent*:

(i) *E has type* 2;

(ii) *S_E has type* 2.

(b) *If E is p-convex (where $1 < p \leqq 2$) and is q-concave for some $2 \leqq q < \infty$ then S_E has type p.*

PROOF. If S_E has type 2 then E has type 2, and if E has type 2 then E is 2-convex and q-concave for some $2 \leqq q < \infty$ ([8] proposition 1.f.7). It is therefore sufficient to establish (b). E' is p'-concave (where $p' = p/(p-1)$) and r-convex for some $1 < r \leqq 2$ ([8] proposition 1.d.4). This means that E' is θ-Hilbertian ([10] theorem 2.2) and so $S_{E'}$ is θ-Hilbertian ([1] theorem 2.4). Thus $S_{E'}$ is K-convex ([11] théorème 1). As $S_{E'}$ has cotype p', it follows that $S_E = (S_{E'})'$ has type p.

Theorem 4 suggests the following problems:

PROBLEM 1. If $(E, \| \quad \|)$ is a q-concave symmetric sequence space (where $2 \leqq q < \infty$), is S_E q-uniformly PL-convex?

PROBLEM 2. If $(E, \| \quad \|)$ is a symmetric sequence space of cotype q (where $2 < q < \infty$), is S_E of cotype q?

Note that it follows from Theorem 4(b) and [8] corollary 1.f.9 that, under the hypotheses of Problem 2, S_E is of cotype s for all $q < s < \infty$.

6. The uniform convexity and uniform smoothness of operator ideals

In this final section we shall prove the following.

THEOREM 5. *Suppose that* $1 < p \leqq q < \infty$ *and that* $(E, \|\cdot\|)$ *is a symmetric Banach sequence space which satisfies an upper* p*-estimate and a lower* q*-estimate. Let* $1 < r < p$, $q < s < \infty$ *and* $r \leqq 2 \leqq s$. *Then there is an equivalent symmetric sequence space norm on* E *such that* S_E *is uniformly convex with modulus of convexity of power type* s *and uniformly smooth with modulus of smoothness of power type* r.

Theorem 5 is a direct consequence of Theorem 2 and the renorming theorem below. The new norm that we introduce involves K_2-functionals, but is rather different from the norms that we considered in earlier sections.

THEOREM 6. *Suppose that* $1 < r < p \leqq q < s < \infty$, *that* $r \leqq 2 \leqq s$ *and that* $(E, \|.\|)$ *is a symmetric Banach sequence space which satisfies an upper* p*-estimate and a lower* q*-estimate. Then there is an equivalent symmetric sequence space norm on* E *which is* K-(r,s)*-monotonic.*

PROOF. By [8] theorem 1.f.7 and proposition 1.d.8 we can give E an equivalent norm $\|.\|_E$ for which $M^{(r)} = M_{(s)} = 1$, and it is clear that this is a symmetric sequence norm.

For each $n = 0, 1, \ldots$ let us set

$$f_n = \sum_{m=2^n}^{2^{n+1}-1} e_m,$$

where e_m is the mth unit vector $(0, \ldots, 0, 1, 0, \ldots)$. If $x \in E$ we set

$$\|x\|_K = \left\| \sum_{n=0}^{\infty} 2^{-n/r} K_2(2^{n\alpha}, x; l_r, l_s) f_n \right\|_E,$$

where $\alpha = 1/r - 1/s$. First we show that there are positive constants a and b such that

$$a\|x\|_K \leqq \|x\|_E \leqq b\|x\|_K$$

for each $x \in E$. We can clearly suppose that $x = x^*$. By Proposition 1, there exist positive constants c_1 and c_2 such that

$$c_2 K_2(2^{n\alpha}, x; l_2, l_s) \leqq \left(\sum_{j=1}^{2^n} x_j^r \right)^{1/r} + 2^{n\alpha} \left(\sum_{j=2^n+1}^{\infty} x_j^s \right)^{1/s}$$
$$\leqq c_1 K_2(2^{n\alpha}, x; l_r, l_s),$$

for each $x \in l_s$. Thus

$$\begin{aligned} c_1\|x\|_K &\geqq \left\| \sum_{n=0}^{\infty} 2^{-n/r}\left(\sum_{j=1}^{2^n} x_j^r\right)^{1/r} f_n \right\|_E \\ &\geqq \left\| \sum_{n=0}^{\infty} x_{2^n} f_n \right\|_E \\ &\geqq \left\| \sum_{n=1}^{\infty} x_n e_n \right\|_E = \|x\|_E, \end{aligned}$$

since $x = x^*$. Conversely,

$$c_2\|x\|_K \leqq \left\| \sum_{n=0}^{\infty} 2^{-n/r}\left(\sum_{j=1}^{2^n} x_j^r\right)^{1/r} f_n \right\|_E + \left\| \sum_{n=0}^{\infty} 2^{-n/s}\left(\sum_{j=2^n+1}^{\infty} x_j^s\right)^{1/s} f_n \right\|_E.$$

Now

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} 2^{-n/r}\left(\sum_{j=1}^{2^n} x_j^r\right)^{1/r} f_n \right\|_E &\leqq \left\| \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} 2^{k-n} x_{2^k}^r\right)^{1/r} f_n \right\|_E \\ &= \left\| \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} 2^{-j} x_{2^{n-j}}^r\right)^{1/r} f_n \right\|_E, \end{aligned}$$

where we set $x_{2^i} = 0$ if $i < 0$. We can write

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} 2^{-j} x_{2^{n-j}}^r\right)^{1/r} f_n = \left(\sum_{j=0}^{\infty} y_j^r\right)^{1/r},$$

where $y_j = 2^{-j/r} \sum_{n=0}^{\infty} x_{2^{n-j}} f_n$, so that by r-convexity

$$\begin{aligned} \left\| \left(\sum_{j=0}^{\infty} y_j^r\right)^{1/r} \right\|_E &\leqq \left(\sum_{j=0}^{\infty} \|y_j\|_E^r\right)^{1/r} \\ &= \left(\sum_{j=0}^{\infty} 2^{-j} \left\| \sum_{n=0}^{\infty} x_{2^{n-j}} f_n \right\|_E^r\right)^{1/r}. \end{aligned}$$

Since $(E, \|\cdot\|_E)$ satisfies an upper p-estimate, its upper Boyd index is greater than or equal to p (cf. [8] p. 132). Therefore there exists a constant M such that

$$\left\| \sum_{n=0}^{\infty} x_{2^{n-j}} f_n \right\|_E \leqq M 2^{j/p} \|x\|_E$$

for all $x \in E$. Thus

$$\left\| \sum_{n=0}^{\infty} 2^{-n/r}\left(\sum_{j=1}^{2^n} x_j^r\right)^{1/r} f_n \right\|_E \leqq M\left(\sum_{j=0}^{\infty} 2^{-j(1-r/p)}\right)^{1/r} \|x\|_E.$$

On the other hand,

$$\left\| \sum_{n=0}^{\infty} 2^{-n/s} \left(\sum_{j=2^n+1}^{\infty} x_j^s \right)^{1/s} f_n \right\|_E \leqq \left\| \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} 2^{k-n} x_{2^k}^s \right)^{1/s} f_n \right\|_E$$

$$\leqq \left\| \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} 2^j x_{2^{j+n}}^s \right)^{1/s} f_n \right\|_E$$

$$= \left\| \left(\sum_{j=0}^{\infty} z_j^s \right)^{1/s} \right\|_E \leqq \left\| \left(\sum_{j=0}^{\infty} z_j^r \right)^{1/r} \right\|_E$$

where $z_j = 2^{j/s} \sum_{n=0}^{\infty} x_{2^{j+n}} f_n$. By r-convexity,

$$\left\| \left(\sum_{j=0}^{\infty} z_j^r \right)^{1/r} \right\|_E \leqq \left(\sum_{j=0}^{\infty} \| z_j \|_E^r \right)^{1/r}$$

$$= \left(\sum_{j=0}^{\infty} 2^{jr/s} \left\| \sum_{n=0}^{\infty} x_{2^{j+n}} f_n \right\|_E^r \right)^{1/r}.$$

Since $(E, \|.\|_E)$ satisfies a lower q-estimate, its lower Boyd index is smaller than or equal to q (cf. [8] p. 132). It follows that there exists a constant $\tilde{M}$ such that

$$\left\| \sum_{n=0}^{\infty} x_{2^{n+j}} f_n \right\|_E \leqq \tilde{M} 2^{-j/q} \| x \|_E,$$

for all $x \in E$. Therefore

$$\left\| \sum_{n=0}^{\infty} 2^{-n/s} \left(\sum_{j=2^n+1}^{\infty} x_j^s \right)^{1/s} f_n \right\|_E \leqq \tilde{M} \left(\sum_{j=0}^{\infty} 2^{j(r/s - r/q)} \right)^{1/r} \| x \|_E.$$

This establishes (†).

The fact that $\|.\|_K$ is a norm follows directly from the triangle inequality for K_2-functionals. The K-(r,s)-monotonicity of $\|.\|_K$ follows directly from the definition of $\|.\|_K$ and the fact that the r-convexity and s-concavity constants of $(E, \|.\|_E)$ are both equal to one.

Theorem 5 has the following consequence.

Corollary 1. *Suppose that $(E, \|.\|)$ is a uniformly convex and uniformly smooth symmetric Banach sequence space with modulus of convexity of power type q (where $2 \leqq q < \infty$) and modulus of smoothness of power type p (where $1 < p \leqq 2$). Then for every $1 < r < p$, $q < s < \infty$, there is an equivalent symmetric sequence space norm on E such that S_E is uniformly convex, with modulus of convexity of power type s, and uniformly smooth with modulus of smoothness of power type r.*

This corollary should be compared with the observation of G. Pisier [10] section 4, (*) (see also [2]) which states that if $1<p\leqq 2\leqq q<\infty$ and E is a symmetric Banach sequence space which is p-convex and q-concave with $M^{(p)}(E)=M_{(q)}(E)=1$ then

$$\left(\tfrac{1}{2}(\|A+B\|_E^{r'}+\|A-B\|_E^{r'})\right)^{1/r'}\leqq(\|A\|_E^r+\|B\|_E^r)^{1/r}$$

for all operators A and B in S_E, where $q'=q(q-1)$, $r=\min(p,q')$ and $r'=r/(r-1)$. From this it follows that S_E is uniformly convex with modulus of convexity of power type r' and uniformly smooth with modulus of smoothness of power type r.

In fact it is possible to improve this last result a little. Suppose that $p<q'$. Define t, s and θ by

$$\frac{1}{t}=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{p'}\right),\quad \frac{1}{p}=\frac{\theta}{s}+(1-\theta)\quad\text{and}\quad\frac{1}{q}=\frac{\theta}{s}\quad(\text{where } p'=p/(p-1)).$$

Then by remark 2.6 of [10] there exists a symmetric Banach sequence space E_0 such that E is the complex interpolation space $[E_0,l_s]_\theta$. As $p<q'$, $1<s<2$ and so S_s is uniformly convex with modulus of convexity of power type 2. This means that there exists $0<c\leqq 1$ such that

$$(\|A+B\|_s^2+c^2\|A-B\|_s^2)^{1/2}\leqq\sqrt{2}(\|A\|_s^2+\|B\|_s^2)^{1/2}$$

for all A and B in S_s, and so the map

$$(A,B)\to(A+B,c(A-B)):S_s\oplus_2 S_s\to S_s\oplus_2 S_s$$

has norm at most $\sqrt{2}$. The map

$$(A,B)\to(A+B,c(A-B)):S_{E_0}\oplus_\infty S_{E_0}\to S_{E_0}\oplus_\infty S_{E_0}$$

has norm 2 and so, since

$$S_E\oplus_t S_E=[S_{E_0}\oplus_\infty S_{E_0},\ S_s\oplus_2 S_s]_\theta,$$

$$(\|A+B\|_E^t+c^t\|A-B\|_E^t)^{1/t}\leqq 2^{1-1/t}(\|A\|_E^t+\|B\|_E^t)^{1/t}$$

for all A and B in S_E. This means that S_E is uniformly convex with modulus of convexity of power type t.

By duality, if $p>q'$ then S_E is uniformly smooth with modulus of smoothness of power type $t'=t/(t-1)$.

The results of this section raise the following natural problems:

PROBLEM 3. Suppose that $1<p\leqq 2\leqq q<\infty$ and that $(E,\|\ \ \|)$ is a symmetric sequence space which is p-convex and q-concave. Is there an equivalent K-(p,q)-monotonic norm on E?

PROBLEM 4. Under the hypotheses of Problem 3, is there an equivalent norm on $(E, \| \ \|)$ such that S_E is uniformly convex with modulus of convexity of power type p and uniformly smooth with modulus of smoothness of power type q?

PROBLEM 5. Suppose that $1 < p \leqq 2 \leqq q < \infty$ and that $(E, \| \ \|)$ is a symmetric sequence space with $M^{(p)}(E) = M_{(q)}(E) = 1$. Is S_E uniformly convex with modulus of convexity of power type p and uniformly smooth with modulus of smoothness of power type q?

PROBLEM 6. Does the conclusion of Problem 4 (or the conclusion of Problem 5) hold if $(E, \| \ \|)$ is uniformly convex with modulus of convexity of power type p and uniformly smooth with modulus of smoothness of power type q?

PROBLEM 7. Does the conclusion of the corollary hold without renorming?

Finally let us mention a consequence of Theorem 6, which may be of independent interest in interpolation theory. For simplicity we formulate it for symmetric Banach sequence spaces; it is also true for rearrangement-invariant function spaces.

COROLLARY 2. *Suppose that $1 < r < p \leqq q < s < \infty$, that $r \leqq 2 \leqq s$, and that $(E, \|.\|)$ is a symmetric Banach sequence space which satisfies an upper p-estimate and a lower q-estimate. Suppose further that T is an operator which is continuous as an operator from $(l_r \oplus l_r)_r$ to $(l_r \oplus l_r)_s$ and from $(l_s \oplus l_s)_r$ to $(l_s \oplus l_s)_s$. Then T is continuous as an operator from $(E \oplus E)_r$ to $(E \oplus E)_s$ and*

$$\| T : (E \oplus E)_r \to (E \oplus E)_s \|$$
$$\leqq c \max[\| T : (l_r \oplus l_r)_r \to (l_r \oplus l_r)_s \|, \| T : (l_s \oplus l_s)_r \to (l_s \oplus l_s)_s \|].$$

Moreover, there exists an equivalent symmetric sequence space norm on E such that the inequality holds with the constant $c = 1$.

SKETCH OF PROOF. By Theorem 6 it is clearly enough to show that if $\|.\|$ is K-(r,s)-monotonic norm on E and T is an operator such that

$$\max[\| T : (l_r \oplus l_r)_r \to (l_r \oplus l_r)_s \|, \| T : (l_s \oplus l_s)_r \to (l_s \oplus l_s)_s \|] \leqq 1$$

then $\| T : (E \oplus E)_r \to (E \oplus E)_s \| \leqq 1$.

Suppose that $(x, y) \in E \oplus E$ and that $x = x_0 + x_1$ and $y = y_0 + y_1$, where x_0 and y_0 are in l_r and x_1 and y_1 are in l_s. Let $(v, z) = T(x, y)$, $(v_0, z_0) = T(x_0, y_0)$ and $(v_1, z_1) = T(x_1, y_1)$. Then v_0 and z_0 are in l_r, v_1 and z_1 in l_s; moreover, $v = v_0 + v_1$ and $z = z_0 + z_1$. Therefore

$$\begin{aligned}(K_2(\tau; v; l_r, l_s)^s + K_2(\tau; z; l_r, l_s)^s)^{1/s} &\leqq \{(\| v_0 \|_r^2 + \tau^2 \| v_1 \|_s^2)^{s/2} + (\| z_0 \|_r^2 + \tau^2 \| z_1 \|_s^2)^{s/2}\}^{1/s} \\ &\leqq \{(\| v_0 \|_r^s + \| z_0 \|_r^s)^{2/s} + \tau^2 (\| v_1 \|_s^s + \| z_1 \|_s^s)^{2/s}\}^{1/2} \\ &= \{\| T(x_0, y_0) \|_{(l_r \oplus l_r)_s}^2 + \tau^2 \| T(x_1, y_1) \|_{(l_s \oplus l_s)_s}^2\}^{1/2} \\ &\leqq \{\| (x_0, y_0) \|_{(l_r \oplus l_r)_r}^2 + \tau^2 \| (x_1, y_1) \|_{(l_s \oplus l_s)_r}^2\}^{1/2} \\ &\leqq \{(\| x_0 \|_r^2 + \tau^2 \| x_1 \|_s^2)^{r/2} + (\| y_0 \|_r^2 + \tau^2 \| y_1 \|_s^2)^{r/2}\}^{1/r}.\end{aligned}$$

Thus, taking the infimum over all such decompositions for x and y, it follows that

$$(K_2(\tau, v; l_r, l_s)^s + K_2(\tau, z; l_r, l_s)^s)^{1/s} \leqq (K_2(\tau, x; l_r, l_s)^r + K_2(\tau, y; l_r, l_s)^r)^{1/r}.$$

The K-(r, s)-monotonicity of the norm implies that

$$\begin{aligned}\| T(x, y) \|_{(E \oplus E)_s} &= (\| v \|^s + \| z \|^s)^{1/s} \\ &\leqq (\| x \|^r + \| y \|^r)^{1/r} = \| (x, y) \|_{(E \oplus E)_r}.\end{aligned}$$

This shows that $\| T : (E \oplus E)_r \to (E \oplus E)_s \| \leqq 1$.

Added in proof. The second named author has recently showed that Problems 1 and 5, hence also Problem 4, have positive solutions.

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